Strong-Coupling Gauge Theory of Nodal Spinons and Antiferromagnetic Long-Range Order

Ikuo Ichinose

Department of Applied Physics, Nagoya Institute of Technology, Nagoya, 466-8555 Japan

Abstract

In this paper we shall study a gauge theory of nodal spinons which appears as a low-energy effective theory for antiferromagnetic (AF) Heisenberg models. In most of studies on the nodal spinons given so far, the gauge interaction between spinons was assumed weak and nonperturbative effects like instantons and vortices were ignored. In the present paper, we shall study strong-coupling gauge theory of nodal spinons and reveal its nontrivial phase structure. To this end, we employ recently developed lattice gauge theory techniques for studying finite-temperature and finite-density gauge theory. At low temperature and low spinon-density region, an AF long-range order exists. As temperature and/or density of spinons are increases, a phase transition to nonmagnetic phase takes place. Order of the phase transition is of second (first) order for low (high) density region of spinons. At a quantum critical point at vanishing temperature $T = 0$, abrupt change of spinon density occurs as a function of the chemical potential. Implications of the results to the heavy-fermion materials and the high-T_c cuprates are discussed.

1 Introduction

Since the discovery of the high-T_c cuprates, search for nontrivial quantum spin liquid for antiferromagnetic (AF) materials is one of the most challenging topics in the condensed matter physics. Among them, possibility of appearance of spin $\frac{1}{2}$ fermionic or bosonic quasi-particles called spinons has been studied from various point of view. In certain class of the AF Heisenberg models in two dimensions (2D), by using slave-particle representation for the quantum spin operators and mean-field theory (MFT) like approximation of decoupling fields, one can simple show that spinons with a relativistic dispersion appear as low-energy excitations[1]. If the fluctuations of the decoupling field are ignored, spin-spin correlation functions exhibit a power-law decay. This type of spin states were dubbed algebraic spin liquids (ASL)[2, 3].

In the ASL, besides spinons, there exist gapless gauge-field excitations, which are phase degrees of freedom of the decoupling field and its fluctuations may destabilize the MF picture of the ASL. It is known that the gauge-field effects in 2D is very strong and low-energy excitations are often drastically influenced by the gauge interactions. Generally speaking there are two phases in the gauge theory; one is confinement phase in which the gauge field fluctuates very strong and charged particles by themselves cannot exist as low-energy excitations. The other is deconfinement phase (it is furthermore classified into Coulomb and Higgs phases) in which nonperturbative effects of the gauge-field fluctuations can be neglected and charged particles and gauge boson itself are low-energy excitations. In the discussion of the ASL, the deconfinement phase of the U(1) gauge field is assumed. Though there is no Maxwell term (or no plaquette term) of the gauge field in the original action, this assumption is justified if the number of gapless spinons N_s is very large because of the shielding effects of the gapless matters.[1]. However our recent study on the deconfinement phase transition by a large number of gapless matter fields indicates that the critical value of N_s is rather large. In Ref.[7], we numerically investigated multi-flavor CP^1 models on a lattice and found that the critical number of two-component complex bosons for the deconfinement transition is 14 which is larger than the number of the two-component Dirac fermions appearing in the typical ASL's. If the gauge theory of nodal spinons is in a confinement phase, weakly interacting spinons may be a good picture in the distance scale of the original lattice spacing but the groundstate and picture of the quasi-excitations at long distances are drastically changed by the violent fluctuations of the gauge field.

In this paper, we shall study the strong-coupling gauge theory of nodal spinons from the above point of view. We shall clarify its phase structure and quasi-excitations. To this end, we employ

[1]For discussions on the problem of deconfinement phenomenon in gauge theories coupled with gapless matter fields, see Refs.[4, 5, 6]

recently developed lattice gauge-theory techniques for studying finite-temperature and finite-density gauge theory[8].

This paper is organized as follows. In Sec.2, we briefly review the ASL in which relativistic Dirac fermions appear. In Sec.3, we introduce a lattice model describing strong-coupling gauge theory of spinons. Symmetries of the model is also discussed. In Sec.4, an effective action is derived by integrating over the gauge field and spinons. In Sec.5, by using a MFT like approximation we clarified the phase structure of the system. Section 6 is devoted for discussion and conclusion.

2 Algebraic Spin Liquid

In this section we shall briefly review the ASL. Let us consider $S = 1/2$ Heisenberg model on the square lattice,

$$H_{\rm AF} = J \sum_{x,i} \vec{S}_x \cdot \vec{S}_{x+i} + \cdots, \tag{2.1}$$

where $\vec{S}_x$ is the spin operator at site $x = (x_1, x_2)$, i is the direction index (sometimes it also denotes the unit vector in the i-th direction), and $J > 0$ is the nearest-neighbor (NN) exchange coupling. In the Hamiltonian (2.1), there may be other interaction terms between spins like the next-NN exchanges, and the ring exchanges, etc which are represented by the ellipsis. The spin operator $\vec{S}_x$ can be represented by the two-component fermionic spinon operator $f_x = (f_{x,1}, f_{x,2}) = (f_{x,\uparrow}, f_{x,\downarrow})$ as,

$$\vec{S}_x = \frac{1}{2} f_x^\dagger \vec{\sigma} f_x, \tag{2.2}$$

where $\vec{\sigma}$ is the Pauli spin matrices and f_x must satisfy the local constraint $\sum_{\alpha=1}^{2} f_{x\alpha}^\dagger f_{x\alpha} = 1$ because the magnitude of the quantum spin is 1/2. By substituting Eq.(2.2) into Eq.(2.1) and decoupling the quartic terms of f_x in $H_{\rm AF}$ by introducing decoupling fields D_{xi}, the decoupled Hamiltonian is obtained as follows,

$$H_{\rm dc} = -\sum_{xi} D_{xi} f_{x+i}^\dagger f_x + \text{H.c.} + |D_{xi}|^2/J + \cdots. \tag{2.3}$$

From Eq.(2.3), it is obvious that the phase of D_{xi}, $iA_{xi} \equiv \log(D_{xi}/|D_{xi}|)$, behaves like a U(1) gauge field under a gauge transformation,

$$f_x \to e^{i\alpha_x} f_x, \quad D_{xi} \to e^{i\alpha_{x+i}} D_{xi} e^{-i\alpha_x}. \tag{2.4}$$

In the MF approximation, the fluctuations of D_{xi} are totally ignored and D_{xi} is replaced with its expectation value $D_{xi}^0 \equiv \langle D_{xi} \rangle$ in Eq.(2.3).

Low-energy excitations are determined by the pattern and symmetry of the MF D_{xi}^0. For example in the π-flux state, phase of the product of four D_{xi}'s around the smallest plaquettes is π. In most of

general flux states, the low-energy excitations are described by two-component (or four-component) relativistic Dirac fields ψ^a where $a = 1, \cdots, n_f$ is the flavor index (the number of nodes). For the π-flux state, $H_{\rm dc}$ in Eq.(2.3) is nothing but the Susskind lattice fermion and n_f is the number of the species doublers ($n_f = 2$). Restoring the local gauge invariance by A_i and imposing local constraint $f_x^\dagger f_x = 1$ by introducing a gauge field A_0, an effective theory for low-energy excitations is given as follows in the continuum,

$$S_{\rm spinon} = \int d^3x \bar{\psi} \Big[\gamma_\mu (\partial_\mu + iA_\mu) \Big] \psi + \cdots, \tag{2.5}$$

where the notations are standard and the ellipsis in Eq.(2.5) represents higher-derivative terms. We have set $\hbar = 1$ and also the "speed of light" (the speed of ψ-quanta) $v_s = 1$. The original AF Heisenberg model is the half-filled state of f_x because the number of f_x is unity at each site. In the effective field theory of the Dirac fermions, *this half-filled state just corresponds to the Dirac sea* in which all the states with negative energy are filled.

Besides the original spin SU(2) symmetry, the continuum action $S_{\rm spinon}$ has the relativistic invariance as well as the flavor symmetry. It is interesting to see how the original spin operators are expressed in terms of the Dirac fermions. From the discussion in Refs.[9, 3], the Néel vector $\vec{N}_x \equiv (-)^{x_1+x_2} \vec{S}_x$ is related with ψ as

$$(-)^{x_1+x_2} \vec{S}_x \propto \bar{\psi} \vec{\sigma} \psi, \tag{2.6}$$

where $\vec{\sigma}$ is the Pauli spin matrix acting on the spin index.

If nonperturbative effects of the gauge field A_μ are negligible and the system (2.5) is in a deconfinement phase of the gauge dynamics, the system can be studied, e.g., by the $1/n_f$ expansion. Composite operators of ψ may acquire an anomalous dimension, but the massless Dirac fermions survive as low-energy excitations at long distances. However if the system is in a confinement phase, the structure of the groundstate and low-energy excitations are drastically influenced by A_μ. For 2D case (and probably also for 3D strong-coupling case), it is expected that for sufficiently large n_f the gauge system is in a deconfinement phase even for the strong-coupling limit without the Maxwell term, whereas it is in a confinement phase for small n_f. That is, there exists a critical number of the massless field separating confinement and deconfinement phases. Very recently we estimated this critical number n_f^c for massless two-component bosonic systems by investigating multi-flavor CP^1 models in $(2+1)$D and obtained $n_f^c = 14$, which is rather large compared to the spinon number in the typical ASL's. Then it is interesting to see how the groundstate and the low-energy excitations of the nodal spinons are changed if the nonperturbative effects like instanton condensation occurs and the gauge dynamics is in a confinement phase. We address this problem in the following sections.

3 Lattice Gauge Theory of Spinons

In this section, we shall study the gauge system (2.5) assuming that the gauge dynamics is in the confinement phase, i.e., $n_f < n_f^c$. We consider the system at finite-temperature (T) and with finite density of spinons. Motivation to consider the finite-density system comes from not only the hole-doped high-T_c cuprates but also from the physics of the heavy-fermion materials. The Hamiltonian of a Kondo-Heisenberg model for the heavy fermions is given as,

$$H_{\rm KH} = \sum_k \epsilon_k c^\dagger_{k\alpha} c_{k\alpha} + J_K \sum_x \vec{S}_x \cdot c^\dagger_x \vec{\sigma} c_x + H_{\rm AF}, \tag{3.1}$$

where $c_{k\alpha}$ represents the conduction electrons with the energy ϵ_k, and J_K is the Kondo coupling. The Kondo coupling prefers hybridization of the conduction electron c_x and the localized spin f_x, whereas the AF coupling tends to make the AF order for $\vec{S}_x$. In the real experiments, external conditions like an external pressure enhances the itinerant properties of the (localized) electrons. This fact implies that the external pressure changes density of states and effectively increases the Kondo coupling J_K and as a result the number of electrons f_x making a spin-singlet pair with the conduction electrons is also increased. (See Fig.1) In the following discussion, the density of spinons is controlled by introducing the chemical potential μ_s instead of considering the Kondo coupling explicitly.

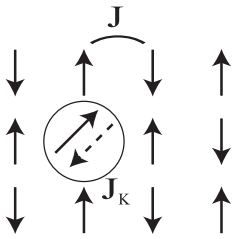

Figure 1: Hybridization of the localized spin and the conduction electron in the AF background of the localized spins. Spin-singlet state of f_x (a solid arrow) and conduction electron (a dashed arrow) appears as a result of the Kondo coupling J_K.

In order to study the strong-coupling gauge theory for the gauge system (2.5), we introduce a d-dimensional *spacetime* hypercubic lattice. The lattice sites are denoted again by x, but x does not represent individual atoms of the original AF Heisenberg model; rather we are considering "coarse grained" lattice model valid at long length scales where the gauge-field fluctuations are very strong.

Lattice model is obtained by replacing derivative with lattice difference in the action (2.5),

$$S_\psi = \frac{1}{2}\sum_{x,\mu}\Big[\bar{\psi}_x\gamma_\mu U_\mu(x)\psi_{x+\mu} - \bar{\psi}_{x+\mu}\gamma_\mu U_\mu^\dagger(x)\psi_x\Big], \tag{3.2}$$

where hereafter $\mu = 0, \cdots, d$ and $U_\mu(x)$ is the compact U(1) gauge field defined on the links $(x, x+\mu)$. However as well-known in the lattice gauge theory, the action S_ψ contains additional low-energy modes besides $k_\mu \sim 0$, the species doublers. In order to decrease these extra low-energy modes and to make the symmetry of the system clear, we make the following change of variables,

$$\psi_x = T(x)\chi_x, \;\; \bar{\psi}_x = \bar{\chi}_x T^\dagger(x), \;\; T(x) = (\gamma_0)^{x_0}\cdots(\gamma_d)^{x_d}. \tag{3.3}$$

Substituting Eq.(3.3) into the action (3.2), we obtain

$$S_\psi = \frac{1}{2}\sum_{x,\mu}\Big[\eta_\mu(x)\bar{\chi}_x U_\mu(x)\chi_{x+\mu} - \eta_\mu(x)\bar{\chi}_{x+\mu}U_\mu^\dagger(x)\chi_x\Big], \tag{3.4}$$

where $\eta_0(x) = 1$ and $\eta_j(x) \equiv (-)^{x_0+\cdots+x_{j-1}}$. From the action (3.4), it is obvious that the Dirac spinor indices are *diagonal* in the χ_x representation, and then we can decrease the number of the low-energy excitations by reducing the number of spinor indices. Hereafter let us put the "flavor number" of χ_x as N_f (i.e., $\chi_x^a; a = 1, \cdots, N_f$). In the continuum, the above system (3.4) contains $2^{[(d+1)/2]}\cdot N_f$ massless Dirac fermions. Therefore $2^{[(d+1)/2]}\cdot N_f = 2n_f$ =[spin degrees of freedom] $\times$[the number of nodes].

As we explained above, we shall consider the system of a finite-density spinons. Then the action is given as follows by introducing the chemical potential μ_s[10],

$$S_\chi = \frac{1}{2}\sum_{x,\mu}\Big[r_\mu(x)\bar{\chi}_x U_\mu(x)\chi_{x+\mu} - r_\mu^{-1}(x)\bar{\chi}_{x+\mu}U_\mu^\dagger(x)\chi_x\Big], \tag{3.5}$$

where $r_0(x) = e^{\mu_s}$ and $r_j(x) = \eta_j(x)$. Because of the particle-hole (electron-positron) symmetry, the physical results are unchanged under a transformation $\mu_s \to -\mu_s$. We assume $\mu_s < 0$ hereafter for the density of spinons decreases by the hole doping or the Kondo coupling.

Before starting study of the spinon-gauge system, we have to notice the following things,

1. As we consider the strong gauge-coupling limit, there are no plaquette (no Maxwell) terms of $U_\mu(x)$ in the action.

2. As $U_0(x)$ is the Lagrange multiplyer field for the local constraint on the number of χ_x (f_x), we have to introduce another matter field representing the "background charge" to make the total charge vanishing. In real materials, this degrees of freedom correspond to doped holes or spin-singlet pairs made of a conduction electron and $\vec{S}_x$.

The requirement 1 in the above is obvious. For 2, we shall introduce the matter field φ_x^a ($a = 1, \cdots, N_f$) by adding the following terms to the action,

$$S_\varphi = \frac{1}{2}\Big[r_0' \bar{\varphi}_x U_0(x) \varphi_{x+\hat{0}} - (r_0')^{-1} \bar{\varphi}_{x+\hat{0}} U_0^\dagger(x) \varphi_x\Big], \tag{3.6}$$

where $r_0' = e^{\mu_s'}$. From S_φ, it is obvious that φ_x carry the same charge with χ_x with respect the Lagrange multiplyer field $U_0(x)$ and they have no hopping terms and are always staying at site x. For the original AF system with "hole doping", the slave-particle representation is useful, in which a hard-core boson operator b_x is introduced for the "hole state" (or spin-singlet state) and the physical state is restricted to $(f_x^\dagger f_x + b_x^\dagger b_x - 1)|phys\rangle = 0$. See Fig.1. The above φ_x on the coarse grained lattice corresponds to b_x on the original lattice.[2]

In the later calculation of the free energy, we shall see that the chemical potentials μ_s and μ_s' always appear in the linear combination $\mu_s - \mu_s'$ as a result of integration over $U_0(x)$. This means that the summation of densities of χ_x and φ_x is automatically vanishing as required by the above physical-state condition.

From S_χ (3.5), the gauge system has the following global U(N_f)× U(N_f) symmetry;

$$\chi_x \to W_D \chi_x, \;\; \bar{\chi}_x \to \bar{\chi}_x W_D^\dagger, \;\; W_D \in \mathrm{U}(N_f), \tag{3.7}$$

and for $W_A \in \mathrm{U}(N_f)$,

$$\begin{aligned} \chi_x \;\; &\to \;\; \begin{cases} W_A \chi_x, & x \;\; \in \text{even site} \\ W_A^{-1} \chi_x, & x \;\; \in \text{odd site} \end{cases} \\ \bar{\chi}_x \;\; &\to \;\; \begin{cases} \bar{\chi}_x W_A, & x \;\; \in \text{even site} \\ \bar{\chi}_x W_A^{-1}, & x \;\; \in \text{odd site.} \end{cases} \end{aligned} \tag{3.8}$$

It is shown that the "Néel order parameter" $\bar{\psi}_x \vec{\sigma} \psi_x = \bar{\chi}_x \vec{\sigma} \chi_x$ is connected with $\bar{\psi}_x \psi_x = \bar{\chi}_x \chi_x$ by the second symmetry in Eq.(3.8), e.g.,

$$\bar{\chi}_x \chi_x \to \bar{\chi}_x e^{i\frac{\pi}{4}\sigma_j} e^{i\frac{\pi}{4}\sigma_j} \chi_x = i \bar{\chi}_x \sigma_j \chi_x, \tag{3.9}$$

where σ_j's are the Pauli spin matrices acting on the spin indices and they are the generators of SU(2)⊂U(N_f). For the spinon-gauge system in the continuum (2.5), it is shown that $\bar{\psi}_x \vec{\sigma} \psi_x = \bar{\chi}_x \vec{\sigma} \chi_x$ and $\bar{\psi}_x \psi_x = \bar{\chi}_x \chi_x$ has the same scaling dimension to all order of the $1/n_f$ expansion. In the present lattice model, these operators are connected by the explicit symmetry.

In the following section, we shall study the lattice spinon-gauge model in the strong-coupling limit by using recently developed techniques of the lattice gauge theory[8].

[2]More precisely, the flavor number of φ_x sholud be $N_f/2$ for the hole-doped case. In this sense, we are think of the two-channel Kondo-Heisenberg model in which two kinds of spin-singlet pair exist. Qualitative phase structure is the same for the N_f and $N_f/2$ cases.

4 Free Energy of the Lattice Spinon-Gauge System

The partition function Z is given as,

$$Z = \int [D\bar{\chi} D\chi D\bar{\varphi} D\varphi DU_0 DU_j] e^{-S_\chi - S_\varphi}, \tag{4.1}$$

where $j = 1, \cdots, d$. As we are considering the finite-temperature system, we impose the (anti)periodic boundary condition on the field variables. Temperature T is given as $T = (aN_0)^{-1}$ where a is the lattice spacing and N_0 is the number of the temporal sites. Hereafter we simply set $a = 1$. Unit of energy is $\hbar v_s/a$ and will be set unity.

There are no gauge-plaquette terms because we are think of the strong-coupling limit. Then we can perform the integration over the spatial gauge field $U_j(x)$ $(j = 1, \cdots, d)$; the one-link integral. We employ the $1/d$-expansion as in the most of calculations and obtain,

$$\int [DU_j] e^{-\frac{1}{2}\sum_{x,j}[\eta_j(x)\bar{\chi}_x U_j(x)\chi_{x+j} - \eta_j(x)\bar{\chi}_{x+j}U_j^\dagger(x)\chi_x]} = \exp\Big[\sum_{x,y} \mathrm{Tr}(M_x V_M(x,y) M_y)\Big], \tag{4.2}$$

where

$$M_x^{ab} = \bar{\chi}_x^a \chi_x^b, \;\; V_M(x,y) = \frac{1}{8}\sum_{j=1}^{d}(\delta_{x,y+j} + \delta_{x,y-j}). \tag{4.3}$$

Let us introduce auxiliary fields Φ_x^{ab} corresponding to the composite fields M_x^{ab}. To this end, we use the following identity,

$$\begin{aligned} &\exp\Big[\sum_{x,y} \mathrm{Tr}(M_x V_M(x,y) M_y)\Big] \\ &= \int [D\Phi] \exp\Big[-\sum_{x,y}\Big\{\mathrm{Tr}(\Phi_x V_M(x,y)\Phi_y) + 2\mathrm{Tr}(\Phi_x V_M(x,y) M_y)\Big\}\Big]. \end{aligned} \tag{4.4}$$

In the following, we shall calculate the free energy (the effective potential) of Φ_x. Because of the symmetry Eqs.(3.7) and (3.8), we can simply set $\Phi_x^{ab} = \Phi_x \delta^{ab}$. (See for example, Eq.(3.9).) Furthermore we consider homogeneous configuration like $\Phi_x = \Phi$, because it is the lowest-energy configuration. Low-energy excitations of Φ_x will be discussed in Sec.6. Then, each term in Eq.(4.4) becomes as,

$$\sum_{x,y} \mathrm{Tr}(\Phi_x V_M(x,y)\Phi_y) \Rightarrow \sum_x \frac{dN_f}{4}\Phi^2, \tag{4.5}$$

and

$$\sum_{x,y} \mathrm{Tr}(\Phi_x V_M(x,y) M_y) \Rightarrow \frac{d}{4}\Phi \sum_x M_x^{aa} = \frac{d}{4}\Phi \sum_x \bar{\chi}_x \chi_x. \tag{4.6}$$

From Eqs.(4.5) and (4.6), the partition function can be written as

$$Z = \int [DU_0][D\bar{\chi}D\chi][D\bar{\varphi}D\varphi] e^{-S'[U_0,\chi,\Phi] - S_\varphi}, \tag{4.7}$$

where

$$S'[U_0,\chi,\Phi]=\sum_x\Big[\frac{1}{2}\bar\chi_x e^{\mu}U_0(x)\chi_{x+\hat 0}-\frac{1}{2}\bar\chi_{x+\hat 0}e^{-\mu}U_0^\dagger(x)\chi_x+\frac{dN_f}{4}\Phi^2-\frac{d}{2}\Phi\bar\chi_x\chi_x\Big]. \tag{4.8}$$

Next we shall integrate over χ_x and φ_x. As they satisfy the anti-periodic boundary condition in the 0-th direction, we introduce the Fourier transformed fields,

$$\chi_{x_0,\vec x}=\frac{1}{\sqrt{N_0}}\sum_{n=1}^{N_0}e^{ip_nx_0}\tilde\chi(n,\vec x),\ \ \bar\chi_{x_0,\vec x}=\frac{1}{\sqrt{N_0}}\sum_{n=1}^{N_0}e^{-ip_nx_0}\tilde{\bar\chi}(n,\vec x), \tag{4.9}$$

where $p_n=\pi(2n-1)/N_0$ and similarly for φ_x. In order to perform the path-integral over the Fourier transformed fields, we take the Polyakov gauge for $U_0(x)$ as follows,

$$U_0(x_0,\vec x)=e^{i\phi(\vec x)/N_0}\quad\text{for all } x_0. \tag{4.10}$$

After integration over $\tilde\chi(n,\vec x)$ etc, we have

$$Z=\int[D\phi]\prod_{\vec x}\prod_{n=1}^{N_0/2}\Big[\sin^2p_n(\phi)+\Big(\frac{d\Phi}{2}\Big)^2\Big]^{N_f}\Big[\sin^2p'_n(\phi)\Big]^{N_f}\cdot e^{-\frac{dN_f}{4}\sum_x\Phi^2}, \tag{4.11}$$

where

$$p_n(\phi)\equiv p_n+\frac{\phi(\vec x)}{N_0}-i\mu_s,\ \ p'_n(\phi)\equiv p_n+\frac{\phi(\vec x)}{N_0}-i\mu'_s. \tag{4.12}$$

The product $\prod_{n=1}^{N_0/2}$ in Eq.(4.11) can be calculated by using the usual techniques of the finite-temperature field theory,

$$\begin{aligned}Z\ &=\ \int[D\phi]\prod_{\vec x}\Big[\cosh(N_0E)\cdot\cos(\phi(\vec x)-iN_0\mu'_s)\\ &\quad+\frac{1}{2}\cos(2\phi(\vec x)-iN_0(\mu_s-\mu'_s))+\frac{1}{2}\cosh(N_0(\mu_s-\mu'_s))\Big]^{N_f}\cdot e^{-\frac{dN_f}{4}\sum_x\Phi^2},\end{aligned} \tag{4.13}$$

where $E=\mathrm{arcsinh}\Big(\frac{d\Phi}{2}\Big)$.

Finally we have to integrate over $\phi(\vec x)$ in Eq.(4.13) to obtain the free energy $F_{\rm eff}=-\log Z/N_0V_s$, where the temperature $T=1/N_0$ and $V_s=\sum_{\vec x}$. Unfortunately, the explicit form of $F_{\rm eff}$ cannot be obtained for general N_f. For each N_f, we integrated over $\phi(\vec x)$ in Eq.(4.13) and then $F_{\rm eff}$ is obtained as follows for $N_f=2$,

$$F_{\rm eff}=\frac{d}{2}\Phi^2-T\log\Big[\frac{1}{4}\cosh^2\Big(\frac{\mu_s-\mu'_s}{T}\Big)+\frac{1}{2}\cosh^2\Big(\frac{E}{T}\Big)+\frac{1}{8}\Big]. \tag{4.14}$$

For $N_f=4$,

$$\begin{aligned}F_{\rm eff}\ &=\ d\Phi^2-T\log\Big[\frac{1}{16}\cosh^4\Big(\frac{\mu_s-\mu'_s}{T}\Big)+\frac{15}{8}\cosh^2\Big(\frac{\mu_s-\mu'_s}{T}\Big)\cos^2\Big(\frac{E}{T}\Big)\\ &\quad+\frac{3}{8}\cosh^4\Big(\frac{E}{T}\Big)+\frac{3}{32}\cosh^2\Big(\frac{\mu_s-\mu'_s}{T}\Big)+\frac{3}{2^7}\Big].\end{aligned} \tag{4.15}$$

Similar expression is obtained for other values of N_f. As stated above, the chemical potentials always appear in the combination $\mu_s - \mu'_s$ and therefore densities of χ_x and φ_x are automatically related with each other,

$$\rho_\chi = -\frac{\partial F_{\rm eff}}{\partial \mu_s} = \frac{\partial F_{\rm eff}}{\partial \mu'_s} = -\rho_\varphi, \tag{4.16}$$

as required by the original local constraint $f_x^\dagger f_x + b_x^\dagger b_x = 1$. Hereafter we set $\mu_T \equiv -\mu_s + \mu'_s \geq 0$.

In the following section, we shall study the phase structure of the spinon-gauge system by using the free energy obtained in this section. To this end, we shall assume a typical form of $F_{\rm eff}$ instead of investigating some specific cases separately.

5 Phase structure of the spinon-gauge system

As we show in the previous section, the free energy $F_{\rm eff}$ has a rather complicated form. In this section we shall study the phase structure of the spinon-gauge system assuming the following "phenomenological" free energy,

$$F_{\rm eff} = \frac{dN_f}{4}\Phi^2 - T\log\Big[\cosh\Big(\frac{N_f\mu_T}{T}\Big) + B\cosh\Big(\frac{N_f E}{T}\Big) + C\Big], \tag{5.1}$$

where B and C are parameters. For general N_f, there appear more complicated terms as in Eq.(4.15), but the essential feature of the free energy and the phase structure can be described by Eq.(5.1).

First let us consider the small Φ region. For $\Phi \ll 1$,

$$F_{\rm eff} = \frac{dN_f}{4}\Phi^2 - \frac{BdN_f}{8[\cosh(\frac{N_f\mu_T}{T}) + B + C]}\cdot\frac{dN_f\Phi^2}{T} + O(\Phi^4). \tag{5.2}$$

As long as the coefficient of the Φ^4-term is positive, $F_{\rm eff}$ exhibits the second-order phase transition and the chemical potential at the critical points μ_T^c is given as a function of T,

$$\mu_T^c = \frac{T}{N_f}\mathrm{arccosh}\Big[\frac{BdN_f}{2T} - B - C\Big]. \tag{5.3}$$

In the region $\mu_T < \mu_T^c$, the field Φ has a nonvanishing expectation value and the symmetry is spontaneously broken. In the AF spin model, an AF order exists in this region.

In particular from Eq.(5.3), the critical T at $\mu_T = 0$, which corresponds to the pure Heisenberg model without hole doping, obtained as

$$T_N = \frac{BdN_f}{2(1+B+C)}, \tag{5.4}$$

which corresponds to the Néel temperature.

As T is lowered, the coefficient of Φ^4 becomes negative and the phase transition changes from a second-order to first-order one. We call this point tricritical point. The tricritical temperature $T_{\rm tri}$

can be easily obtained from Eq.(5.1) as

$$T_{\rm tri} = \frac{N_f}{2}. \tag{5.5}$$

From Eq.(5.3), the chemical potential at the tricritical point is given as,

$$\mu_T^{\rm tri} = \frac{1}{2} \cdot \mathrm{arccosh}\Big((d-1)B - C\Big). \tag{5.6}$$

If $\frac{dB}{B+C+1} < 1$, the phase transition is everywhere of first order.

Next let us consider the $T = 0$ case; the finite-density *quantum phase transition* (QPT). In this case, the free energy reduces to,

$$F_{\rm eff}[\Phi] = \frac{dN_f}{4}\Phi^2 - N_f \cdot \max\Big\{\mu_T, E[\Phi]\Big\}. \tag{5.7}$$

At $\Phi = 0$, $E[\Phi = 0] = 0$ and therefore $F_{\rm eff}[\Phi = 0] = -N_f\mu_T$. It is obvious that $\Phi = 0$ is a local minimum of $F_{\rm eff}[\Phi]$. There is another local minimum at Φ_0, where Φ_0 is the solution to the equation $\partial F_{\rm eff}[\Phi]/\partial\Phi = 0$ for $E[\Phi] > \mu_T$. Φ_0 is easily obtained as

$$\Phi_0^2 = \frac{2\sqrt{1+d^2} - 2}{d^2}, \tag{5.8}$$

and the local minimum is calculated as

$$F_{\rm eff}[\Phi_0] = \frac{dN_f}{4}\Phi_0^2 - N_f E[\Phi_0] < 0. \tag{5.9}$$

As μ_T increases, the global minimum changes from Φ_0 to $\Phi = 0$. The critical chemical potential is then given by

$$\mu_{T0}^c = E[\Phi_0] - \frac{d}{4}\Phi_0^2, \tag{5.10}$$

and the expectation value of Φ behaves as

$$\langle\Phi\rangle(T=0) = \begin{cases} \Phi_0 & \text{for } \mu_T < \mu_{T0}^c \\ 0 & \text{for } \mu_T > \mu_{T0}^c. \end{cases} \tag{5.11}$$

Similarly, the density of the spinons $\rho \equiv \rho_\chi = -\rho_\varphi$ is calculated from Eq.(5.7),

$$\rho = \frac{\partial F_{\rm eff}}{\partial \mu_T} = \begin{cases} 0 & \text{for } \mu_T < \mu_{T0}^c \\ -N_f & \text{for } \mu_T > \mu_{T0}^c. \end{cases} \tag{5.12}$$

The above result Eq.(5.11) and (5.12) indicates that a strong first-order phase transition from ordered to disordered phases takes place at $T = 0$, and an abrupt change of the density of the spinons from the empty state to the saturated state also occurs there. For $\mu_T > \mu_{T0}^c$, the density of the localized electrons participating in AF magnetism are vanishing i.e., all electrons are itinerant.

Finally let us show numerical calculations for some typical values of the parameters in $F_{\rm eff}$. We put $d = 3,\ B = 1,\ C = 0$ and $N_f = 4$ in Eq.(5.1). In this case, $T_N = 3,\ T_{\rm tri} = 2,\ \mu_T^{\rm tri} = 0.658\cdots,\ \mu_{T0}^c = 0.549\cdots$ and $\Phi_0 = 0.693\cdots$. In Fig.2, we show the phase diagram in the $\mu_T - T$ plane. The spinon-gauge system at the strong-coupling limit has the symmetry-breaking phase in the low-T and low-density region. In the high-T region $T > T_{\rm tri}$, the phase transition is of second-order whereas in the low-T region $T < T_{\rm tri}$ it is of first-order.At quantum critical point $(\mu_T, T) = (\mu_{T0}^c, 0)$, the density of spinons changed drastically from zero to $-N_f$. The phase transition line show a reentrant behavior, but this behavior may be an artifact of the mean-field approximation. See discussion in the following section. In Figs.3 and 4, we show the expectation value of Φ as a function of μ_T. For $T < T_{\rm tri}$, Φ exhibits a discontinuity at $\mu_T = \mu_{T0}^c$. In Figs.5 and 6, we show the density ρ as a function of μ_T.

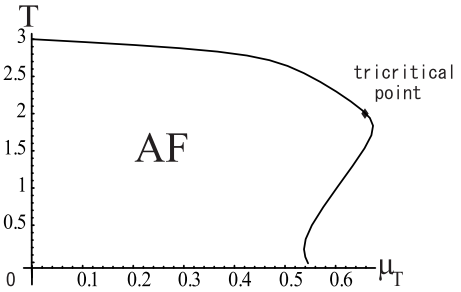

Figure 2: Phase diagram obtained from the free energy by using MFT like approximation. Low-T and low-density region of spinons has a AF long-range order. Symbol • denotes the tricritical point at $\mu_T = \mu_T^{\rm tri}$. Above (Below) $T_{\rm tri}$, the phase transition is of second (first) order.

6 Discussion and conclusion

In this paper we studied the strong-coupling region of the spinon-gauge system on the lattice at finite-T and finite density of spinons. By using the MFT like approximation, we found that at low-T and low density the system has the AF long-range order (LRO). We expect that this LRO is weak magnetism compared with the microscopic local Néel order as explained in Ref.[11] and observed by the experiments of the heavy-fermion systems. Low-energy excitations in the ordered phase are easily identified as follows by the symmetry of the system Eq.(3.8) and also from the result of the

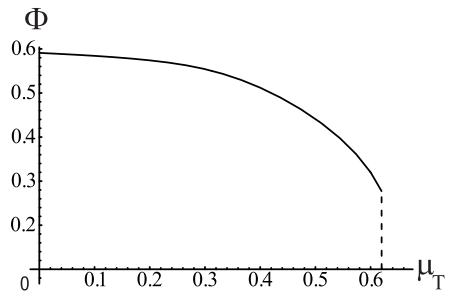

Figure 3: Expectation value Φ as a function of the chemical potential $\mu_T = -\mu_s + \mu'_s$ at $T = 1.5 < T_{\rm tri}$. The phase transition is of first order.

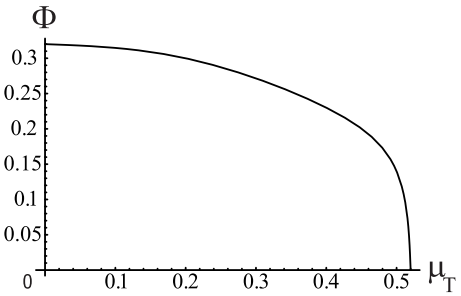

Figure 4: Expectation value Φ as a function of the chemical potential $\mu_T = -\mu_s + \mu'_s$ at $T = 2.5 > T_{\rm tri}$. The phase transition is of second order.

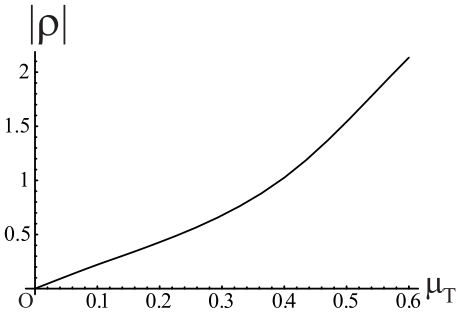

Figure 5: Density of spinons $\rho(< 0)$ as a function of the chemical potential $\mu_T = -\mu_s + \mu_s'$ at $T = 1.5 < T_{\rm tri}$.

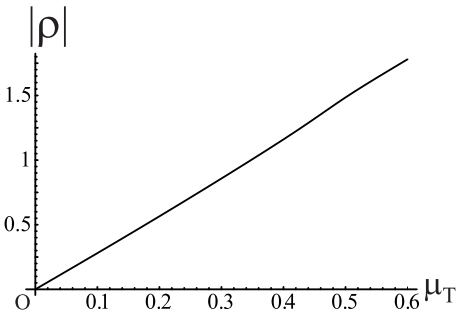

Figure 6: Density of spinons $\rho(< 0)$ as a function of the chemical potential $\mu_T = -\mu_s + \mu_s'$ at $T = 2.5 > T_{\rm tri}$.

one-link integral Eq.(4.2),

$$\Phi_x = \langle\Phi\rangle \cdot \exp\left[i\epsilon(x)\sum_K T^K \pi^K(x)\right], \tag{6.1}$$

where $\epsilon(x) = (-)^{x_0+\cdots+x_{d-1}}$, T^K is the generators of U(N_f) and $\pi^K(x)$ are gapless excitations, the spinwaves. Then one can easily expect that an effective field theory for the ordered phase is a U(N_f) nonlinear σ-model.

At higher-T, the effect of fluctuations of $\pi^K(x)$ is larger and destabilizes the AF order. Then the reentrance of the transition line observed in the MFT like approximation may disappear as a result of the fluctuations described by $\pi^K(x)$.

As discussed rather in detail in Ref.[12], the MFT-like approximation is reliable for small N_f, whereas for large N_f an instability of the state with long-range order to a symmetric state appears. Therefore the strong-coupling gauge theory and also the MFT-like approximation employed in the present paper is justified only for small N_f cases.

In this paper we considered the gauge theory of spinons. It is quite interesting to study a coupled system of the conduction electrons and spin degrees of freedom and discuss the quantum phase transition to heavy-fermion state[11].

References

[1] J.B.Marston and I.Affleck, Phys.Rev.B**39**, 11538(1989).

[2] W.Rantner and X.-G.Wen, Phys.Rev.Lett.**86**, 3871(2001).

[3] M.Hermele, T.Senthil, and M.P.A.Fisher, Phys.Rev.B**72**, 104404(2005).

[4] C.Nayak, Phys.Rev.Lett. **85**, 178(2000).

[5] I.Ichinose and T.Matsui, Phys.Rev.Lett. **86**, 942(2001).

[6] I.Ichinose, T.Matsui and M.Onoda, Phys.Rev.**B64**, 104516(2001).

[7] S.Takashima, I.Ichinose, and T.Matsui, Phys.Rev.B**73**, 075119(2006).

[8] See for example, Y.Nishida, Phys.Rev.D**69**, 094501(2004).

[9] L.Susskind, Phys.Rev.D**16**, 3031(1977).

[10] P.Hansenfratz and F.Karsch, Phys.Lett.B**125**, 308(1983).

[11] See for example, T.Senthil, M.Vojta, and S.Sachdev, Phys.Rev.B**69**, 035111(2004).

[12] I.Ichinose, Mod.Phys.Lett.A**17**, 1355(2002).